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Abstract

A group of experts must choose the winner of a competition. The honest opinions of the experts must be aggregated to determine the deserving winner. The aggregation rule is majoritarian if it respects the honest opinion of the majority of experts. An expert might not want to reveal her honest opinion if, by doing so, a contestant that she likes more is chosen. Then, we have to design a mechanism that implements the aggregation rule. We show that, in general, no majoritarian aggregation rule is Nash implementable, even if no expert has friends or enemies among the contestants.

Key Words: mechanism design; Nash equilibrium; aggregation of experts' opinions; jury.

J.E.L. Classification Numbers: C72, D71, D78.

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1 Introduction

A group of experts must choose the winner of a competition. The different opinions of the experts about who is the best contestant must be aggregated to determine the deserving winner. The aggregation rule is majoritarian if it selects as deserving winner the contestant who is viewed as the best one by a majority of experts whenever such contestant exists. Each expert has a preference relation over the set of contestants that may depend on her honest opinion about who is the best contestant, but it is not necessarily determined by it. For example, an expert might want a friend of her to win, even if she does not believe this contestant is the best one. An expert might not want to reveal her honest opinion if, by doing so, a contestant that she likes more is chosen. Then, the planner has to design a mechanism that gives the incentives to the experts to always choose the contestant that would be considered deserving winner by the aggregation rule according to their honest opinions. When such a mechanism exists, we say that the aggregation rule is implementable. In the present paper, we study the problem of implementing majoritarian aggregation rules when the equilibrium concept used by the experts is Nash equilibrium.

For a majoritarian aggregation rule to be Nash implementable, the group of experts must satisfy some "impartiality" requirements. Amorós (2018) stated a necessary condition for the implementation of any majoritarian aggregation process in any ordinal equilibrium concept.[1] This condition requires that all experts must be impartial with respect to each pair of contestants (an expert is impartial with respect to a pair of contestants if, whenever she believes that one of the two contestants is the best one, then she prefers that contestant to the other). As strong as it is, the fulfillment of this condition does not guarantee the existence of a majoritarian aggregation rule that is implementable. Whether the condition is sufficient or not may depend on the equilibrium concept considered. We show that, in general, no majoritarian aggregation rule is Nash implementable, regardless of the "impartiality" properties that the group of experts may satisfy. Even if all experts are impartial and have no friends or enemies among the contestants, no majoritarian aggregation rule can be implemented in Nash equilibrium.

Amorós (2013) studied a model where all experts have always the same

[1] An equilibrium concept is ordinal if the set of equilibrium messages does not change unless some expert changes her preferences.

opinion (*i.e.*, there is a contestant, known by all experts, who objectively is the best one). In this case, the problem is reduced to designing a mechanism that gives incentives to experts to always choose the best contestant, which is possible under a minimum requirement of impartiality. The present paper is related to the literature on the Condorcet Jury Theorem (e.g., Austen-Smith and Banks, 1996; Dugan and Martinelli, 2001; Feddersen and Pesendorfer; 1998, McLennan, 1998). Unlike this literature, however, our experts may have different opinions and not agree on the overall objective. There are some results in the literature showing the impossibility of Nash implementing social choice functions when the domain of admissible preferences is “rich” (e.g. Saijo, 1987). Our result cannot be deduced from this literature, because a majoritarian aggregation rule does not have to be single-valued.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 presents the result. Section 4 concludes.

2 The model

A finite group of three or more experts $E = \{1, 2, 3...\}$ must choose one winner among a finite group of three or more contestants $N = \{a, b, c, ...\}$. General elements of E are denoted by i, j, etc. and general elements of N are denoted by x, y, etc. Each expert $i \in E$ has an honest opinion about who is the best contestant, $w_i \in N$. Let $w = (w_i)_{i \in E} \in N^{|E|}$ denote a profile of experts' opinions. Let $\Re$ denote the class of all complete, reflexive, and transitive preference relations over N. Each expert i has a preference function $R_i : N \longrightarrow \Re$ that associates with each possible opinion $w_i \in N$ a preference relation $R_i(w_i) \in \Re$. Let $P_i(w_i)$ denote the strict part of $R_i(w_i)$. The fact that i believes w_i is the best contestant does not necessarily imply that w_i is her most preferred contestant. Table 1 shows an example of preference function (contestants ranked higher in the table are strictly preferred to those ranked lower). Let $\mathcal{R}$ denote the class of all possible preference functions.

$R_i : N \longrightarrow \Re$

$w_i =$	a	b	c
	a	a	a
Preferences	bc	b	c
		c	b

Table 1 Example of preference function when $N = \{a, b, c\}$.

Let 2^N_2 denote the set of all possible pairs of contestants. A general pair of contestants is denoted xy. An expert is known to be **impartial** with respect to a pair of contestants if, whenever she believes that one of the two contestants is the best contestant of the competition, she prefers that contestant to the other. Each expert i is characterized by a **set of pairs of contestants with respect to whom she is known to be impartial**, $I_i \subset 2^N_2$.

Definition 1 *A preference function $R_i \in \mathcal{R}$ is **admissible** for expert i at $I_i \subset 2^N_2$ if, for each $xy \in I_i$, $x\ P_i(x)\ y$ and $y\ P_i(y)\ x$.*

For example, the preference function in Table 1 is admissible at $I_i = \{bc\}$, but it is not admissible at $\hat{I}_i = \{ab\}$. Let $\mathcal{R}(I_i)$ denote the class of all preference functions that are admissible for i at I_i. A **jury configuration** is a profile of sets $I = (I_i)_{i\in E}$. A profile of preference functions $R = (R_i)_{i\in E}$ is admissible at I if $R_i \in \mathcal{R}(I_i)$ for every $i \in E$. Let $\mathcal{R}(I) \subset \mathcal{R}^{|E|}$ denote the set of admissible profiles of preference functions at I.

Given a jury configuration I, a **state** is a profile of admissible preference functions together with an admissible profile of expert's opinions, $(R, w) \in \mathcal{R}(I) \times N^{|E|}$. Let 2^N denote the set of all subsets of N. A **social choice rule** (SCR) is a correspondence $F : \mathcal{R}(I) \times N^{|E|} \to 2^N\backslash\{\emptyset\}$. We want the deserving winner to depend only on jurors' honest opinions, not on their preferences. For that reason, we only consider SCRs such that, for every $(R, w), (\hat{R}, w) \in \mathcal{R}(I) \times N^{|E|}$, $F(R, w) = F(\hat{R}, w)$. This is equivalent to defining a SCR as a mapping $F : N^{|E|} \to 2^N\backslash\{\emptyset\}$. Let $\mathcal{F}$ denote the class of all these SCRs. A SCR is majoritarian if it only selects the contestant that is viewed as the best one by more than half of the experts, whenever that contestant exists. For each $w \in N^{|E|}$ and $x \in N$, let $E^x_w = \{i \in E : w_i = x\}$.

Definition 2 *A SCR $F \in \mathcal{F}$ is **majoritarian** if, for every $w \in N^{|E|}$ and $x \in N$ such that $|E^x_w| > \frac{|E|}{2}$, we have $F(w) = x$.*

A **mechanism** is a pair $\Gamma = (M, g)$, where $M = \times_{i\in E} M_i$, M_i is a message space for expert i, and $g : M \to N$ is an outcome function. A profile of messages $m \in M$ is a **Nash equilibrium** of mechanism $\Gamma = (M, g)$ at state (R, w) if for each $i \in E$ and each $\hat{m}_i \in M_i$, $g(m)\ R_i(w_i)\ g(\hat{m}_i, m_{-i})$. Let $N(\Gamma, R, w) \subset M$ denote the set of profiles of messages that are a Nash equilibrium of mechanism Γ at state (R, w). Let $g(N(\Gamma, R, w)) = \{x \in N :$

there is some $m \in N(\Gamma, R, w)$ with $g(m) = x\}$ be the corresponding set of Nash equilibrium outcomes. A mechanism implements a SCR if, in every state, the contestants prescribed by the SCR are selected in equilibrium.

Definition 3 *Given a jury configuration I, a mechanism $\Gamma = (M, g)$* ***implements*** *a SCR $F \in \mathcal{F}$ in Nash equilibrium, if, for each state $(R, w) \in \mathcal{R}(I) \times N^{|E|}$, one has that $g(N(\Gamma, R, w)) = F(w)$.*

3 The results

In principle, a SCR could be Nash implementable or not depending on the jury configuration. The idea is that, the greater the sets of pairs of contestants with respect to which the experts are known to be impartial, the smaller the set of admissible states, which can only facilitate implementation. However, we show that no majoritarian SCR can be implemented in Nash equilibrium, regardless of the properties that the jury configuration may satisfy.

We begin by establishing a necessary and sufficient condition for a majoritarian SCR to be implementable in Nash equilibrium. In our setting, this condition is equivalent to the well-known condition of Maskin monotonicity (Maskin, 1999). Maskin monotonicity together with a no veto power condition (which is trivially satisfied by any majoritarian SCR) are sufficient conditions for Nash implementation when there are at least three agents.

Lemma 1 *Let $F \in \mathcal{F}$ be a SCR implementable in Nash equilibrium. Then, for every $w, \hat{w} \in N^{|E|}$ and $x \in N$, if $x \in F(w)$ and $x \notin F(\hat{w})$, then there exists $i \in E$ such that $w_i = x$ and $w_i\hat{w}_i \in I_i$. Moreover, if F is majoritarian, the previous condition is also sufficient for the Nash implementability of F.*

Proof. First we prove that, given any jury configuration I, a SCR $F \in \mathcal{F}$ satisfies the condition of the statement if and only if it satisfies Maskin monotonicity, a necessary condition for implementation in Nash equilibrium (Maskin, 1999). Maskin monotonicity requires that, for every two states $(R, w), (\hat{R}, \hat{w}) \in \mathcal{R}(I) \times N^{|E|}$ and contestant $x \in N$, if $x \in F(w)$ and $x \notin F(\hat{w})$, then there exist $i \in E$ and $y \in N$ such that $x \; R_i(w_i) \; y$ and $y \; \hat{P}_i(\hat{w}_i) \; x$. Let $(R, w), (\hat{R}, \hat{w}) \in \mathcal{R}(I) \times N^{|E|}$ and $x \in N$ be such that $x \in F(w)$ and $x \notin F(\hat{w})$. From the condition of the statement, there is $i \in E$ be such that $w_i = x$ and $w_i\hat{w}_i \in I_i$. Because $w_i\hat{w}_i \in I_i$ and $R_i, \hat{R}_i \in \mathcal{R}(I_i)$,

$w_i \; R_i(w_i) \; \hat{w}_i$ and $\hat{w}_i \; \hat{P}_i(\hat{w}_i) \; w_i$. Therefore, $x \; R_i(w_i) \; \hat{w}_i$ and $\hat{w}_i \; \hat{P}_i(\hat{w}_i) \; x$, which implies that Maskin monotonicity is satisfied. Now, we prove that if F satisfies Maskin monotonicity, the it satisfies the condition of the statement. Suppose by contradiction that F satisfies Maskin monotonicity but there are $w, \hat{w} \in N^{|E|}$ and $x \in N$ with $x \in F(w)$, $x \notin F(\hat{w})$, and such that, for every $i \in E$ with $w_i = x$, we have $w_i\hat{w}_i \notin I_i$. Note that, for every $i \in E$ with either (i) $w_i \neq x$ or (ii) $w_i = x$ and $w_i\hat{w}_i \notin I_i$, there exists $R_i \in \mathcal{R}(I_i)$ such that, for every $y \in N$, if $x \; R_i(w_i) \; y$ then $x \; R_i(\hat{w}_i) \; y$. Therefore, there exists some $R \in \mathcal{R}(I)$ such that, for every $i \in E$ and every $y \in N$, if $x \; R_i(w_i) \; y$ then $x \; R_i(\hat{w}_i) \; y$. Hence, $(R, w), (R, \hat{w}) \in \mathcal{R}(I) \times N^{|E|}$ are such that $x \in F(w)$, $x \notin F(\hat{w})$, and for every $i \in E$ and every $y \in N$, if $x \; R_i(w_i) \; y$ then $x \; R_i(\hat{w}_i) \; y$. This contradicts that F satisfies Maskin monotonicity. With at least three experts, Maskin monotonicity plus a condition called no veto power is sufficient for implementation in Nash equilibrium (Maskin 1999). No veto power requires that, for every $(R, w) \in \mathcal{R}(I) \times N^{|E|}$, $x \in N$, and $j \in E$, if $x \; R_j(w_j) \; y$ for every $y \in N$ and every $i \in E \backslash \{j\}$ then $x \in F(w)$. Note that if F is majoritarian then it satisfies no veto power. ■

Now we can prove that, unless $|E| = 4$ and $|N| = 3$, there is no majoritarian SCR satisfying the necessary condition for Nash implementation stated in Lemma 1, regardless of how the jury configuration is,

Theorem 1 *Unless $|E| = 4$ and $|N| = 3$, no majoritarian SCR is implementable in Nash equilibrium for any jury configuration.*

Proof. Given any jury configuration I, suppose there exists a majoritarian SCR $F \in \mathcal{F}$ that is implementable in Nash equilibrium. Next we show that, unless $|E| = 4$ and $|N| = 3$, there exist $w, \hat{w} \in N^{|E|}$ and $x \in N$ such that $x \in F(w)$, $x \notin F(\hat{w})$ and, for every $i \in E$ with $w_i = x$, $\hat{w}_i = w_i$. By Lemma 1, this contradicts that F is implementable in Nash equilibrium. We distinguish three different cases.

Case 1. $|E|$ is an odd number.

Suppose $|E|$ is odd. Let $w \in N^{|E|}$ be such that, for some $x, y, z \in N$, $|E_w^x| = \left\lfloor \frac{|E|}{2} \right\rfloor$, $|E_w^y| = \left\lfloor \frac{|E|}{2} \right\rfloor$, and $|E_w^z| = 1$.[2] Let $t \in F(w)$. Suppose first that $t = x$. Let $\hat{w} \in N^{|E|}$ be such that (1) $\hat{w}_i = y$ for every $i \in E$ with $w_i \neq x$, and (2) $\hat{w}_i = w_i$ for every $i \in E$ with $w_i = x$. Note that $|E_{\hat{w}}^y| > \frac{|E|}{2}$. Because

[2] For each $\alpha \in \mathbb{R}$, $\lfloor \alpha \rfloor = \max\{\beta \in \mathbb{Z} : \beta \leq \alpha\}$, where $\mathbb{Z}$ is the set of integers.

F is majoritarian, $F(\hat{w}) = y$. Then, $w, \hat{w} \in N^{|E|}$ are such that $x \in F(w)$ and $x \notin F(\hat{w})$ and, for every $i \in E$ with $w_i = x$, $\hat{w}_i = w_i$. Suppose now that $t \neq x$. Let $\hat{w} \in N^{|E|}$ be such that (1) $\hat{w}_i = x$ for every $i \in E$ with $w_i \neq t$, and (2) $\hat{w}_i = w_i$ for every $i \in E$ with $w_i = t$. Then $|E^x_{\hat{w}}| > \frac{|E|}{2}$ and, because F is majoritarian, $F(\hat{w}) = x$. Then, $t \in F(w)$, $t \notin F(\hat{w})$, and, for every $i \in E$ with $w_i = t$, $\hat{w}_i = w_i$.

Case 2. $|E|$ is an even number and $|E| > 4$.

Suppose $|E| > 4$ is even. Let $w \in N^{|E|}$ be such that, for some $x, y, z \in N$, $|E^x_w| = \frac{|E|}{2} - 1$, $|E^y_w| = \left\lceil \frac{|E|}{4} \right\rceil$, and $E^z_w = \left\lfloor \frac{|E|}{4} \right\rfloor + 1$, then $x \notin F(w)$.[3] Let $t \in F(w)$. Suppose first that $t = x$. Let $\hat{w} \in N^{|E|}$ be such that, for each $i \in E$, (1) if $w_i \neq x$ then $\hat{w}_i = y$, and (2) if $w_i = x$ then $\hat{w}_i = w_i$. Then, $|E^y_{\hat{w}}| > \frac{|E|}{2}$, and because F is majoritarian, $F(\hat{w}) = y$. Therefore, $w, \hat{w} \in N^{|E|}$ are such that $x \in F(w)$ and $x \notin F(\hat{w})$ and, for every $i \in E$ with $w_i = x$, $\hat{w}_i = w_i$. Suppose now that $t \neq x$. Let $\hat{w} \in N^{|E|}$ be such that, for each $i \in E$, (1) if $w_i \neq t$ then $\hat{w}_i = x$, and (2) if $w_i = t$ then $\hat{w}_i = w_i$. Note that $|E^x_{\hat{w}}| > \frac{|E|}{2}$. Because F is majoritarian, $F(\hat{w}) = x$. Then, $w, \hat{w} \in N^{|E|}$ are such that $t \in F(w)$ and $t \notin F(\hat{w})$ and, for every $i \in E$ with $w_i = t$, $\hat{w}_i = w_i$.

Case 3. $|E| = 4$ and $|N| \geq 4$.

Suppose $|E| = 4$ and $|N| \geq 4$. Let $w \in N^{|E|}$ be such that $w_i \neq w_j$ for every $i, j \in E$. Let $x \in F(w)$. Let $\hat{w} \in N^{|E|}$ be such that, for some $y \in N \backslash \{x\}$ and for each $i \in E$, (1) if $w_i \neq x$ then $\hat{w}_i = y$ and (2) if $w_i = x$ then $\hat{w}_i = w_i$. Note that $|E^y_{\hat{w}}| > \frac{|E|}{2}$. Because F is majoritarian, $F(\hat{w}) = y$. Then, $w, \hat{w} \in N^{|E|}$ are such that $x \in F(w)$ and $x \notin F(\hat{w})$ and, for every $i \in E$ with $w_i = x$, $\hat{w}_i = w_i$. ■

Remark 1 *Let $|E| = 4$ and $|N| = 3$. Suppose that $I_i = 2^N_2$ for every $i \in E$ (i.e., all experts are impartial with respect to all pairs of contestants). Let $F \in \mathcal{F}$ be such that, for each $w \in N^{|E|}$, $F(w) = \{x \in N : |E^x_w| \geq \frac{|E|}{2}\}$. Because $|E| = 4$ and $|N| = 3$, $F(w) \neq \emptyset$ for every $w \in N^{|E|}$. Clearly, F is majoritarian. Moreover, for every $w, \hat{w} \in N^{|E|}$ and $x \in N$, if $x \in F(w)$ and $x \notin F(\hat{w})$, then there exists $i \in E$ such that $w_i = x$ and $\hat{w}_i \neq x$. Because $I_i = 2^N_2$, $w_i\hat{w}_i \in I_i$. Then, by Lemma 1, F is implementable in Nash equilibrium.*

[3] For each $\alpha \in \mathbb{R}$, $\lceil \alpha \rceil = \min\{\beta \in \mathbb{Z} : \beta \geq \alpha\}$, where $\mathbb{Z}$ is the set of integers.

4 Conclusion

We have studied the problem of Nash implementing the rule that aggregates the honest opinions of a group of experts about who is the best contestant of a competition. When implementing an aggregation rule, the best scenario would be that no expert has friends or enemies among the contestants. In the present paper, we have shown that no majoritarian aggregation rule can be implemented in Nash equilibrium even if the former condition is satisfied. The question of whether it is possible to implement some majoritarian rule in other different equilibrium concepts is still open.

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