On Counting the Poor in a Multidimensional Context

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Abstract

We claim here that, in the context of multidimensional poverty measurement, defining the set of the poor independently on the poverty measure yields nonsensical results and implies a violation of some basic properties. We show that an index that satisfies monotonicity, focus, and restricted continuity, defines endogenously the set of the poor as the set of agents for which the index shows the presence of poverty when considered as a society of a single individual.

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1 Introduction

The standard approach to poverty measurement consists of building an indicator in two steps. The first one refers to the incidence of poverty and identifies the poor by some criterion that depends on a given reference level. The second step deals with the intensity of poverty and aims at providing a measure of the size of the deprivation experienced by the poor. Step one answers the question of "Who are the poor?" whereas step two responds to "How poor are the poor?" It is understood that a poverty measure should take into account both aspects and also that the intensity should also include distributive considerations (e.g. inequality among the poor).

When there is a single dimension involved, typically income, a person is considered poor if the value of her achievement is below some given threshold and there is no ambiguity about what being poor means. Counting the poor in a multidimensional context is not that simple though, as the poverty threshold is now a vector with \( n \geq 2 \) components. It is clear that an agent whose achievements are above (resp. below) the threshold in every component is non-poor (resp. poor). The problem arises when the agents' achievements may exceed the threshold levels in some dimensions and fall short in some others. There are two extreme positions on this respect, each one with arguments pro and con. On the one hand, there is the union approach that declares poor anyone who is below the reference value in some dimension. On the other hand, we find the intersection approach according to which one person is poor only if all her achievements are simultaneously below the reference values. There is some consensus in that the first approach may overestimate the number of poor and the second one is far too restrictive. That is why there are also some intermediate proposals, as those in Bourguignon and Chakravarty (2003), Alkire and Foster (2007), Lugo and Maasoumi (2008) or Alkire and Santos (2010). Yet there does not seem to be a clear cut principle to decide which intermediate approach is suitable.

The literature on multidimensional poverty has kept the tradition of determining who are the poor as a separate issue of the choice of the poverty index, as in the single dimensional case. This is somehow bizarre, specially when we deal with decomposable or consistent indices, as those measures can be regarded as aggregating individual poverty measures. As a consequence we may find that the a priori chosen set of the poor differs from the set of the poor that would result from applying our poverty measure. This lack of coherence raises some concerns that are not merely esthetical. The purpose of this paper is to show that, in a multidimensional context, we are not free to decide who are the poor independently on the poverty measure, unless we violate some very basic properties. As a consequence, all those indices
that satisfy the standard properties incorporate necessarily an endogenous
determination of the set of the poor.

The rest of the paper goes as follows. We first introduce some nota-
tion and define the key properties that will be used in the discussion. We
show that, when a poverty index satisfies monotonicity, focus and restricted
continuity, it determines the set of the poor endogenously.

2 The model

We consider here a standard model of poverty evaluation in a multidimen-
sional context. Treating poverty as a multidimensional phenomenon becomes
necessary when we cannot properly aggregate the different poverty dimen-
sions by some suitable price system; this may be due to the absence or imper-
fection of markets, the presence of externalities, the non-substitutability of
some dimensions below some levels, the inadequacy of market prices to eval-
uate the consumption of the poor, etc. The reader is referred to the works
and Chakravarty (2003), and Chakravarty (2009) for a detailed discussion of
those models.

2.1 Preliminaries

Let $N = \{1, 2, ..., n\}$ denote a society consisting of $n$ individuals and let
$K = \{1, 2, ..., k\}$ be a set of factors. Each factor corresponds to a variable
that approximates one dimension of poverty deemed relevant. A social state
is a positive matrix $Y = \{y_{ij}\}$ with $n$ rows, one for each individual, and $k$
columns, one for each factor. The entry $y_{ij} \in \mathbb{R}_{++}$ describes the value of fac-
tor $j$ for individual $i$. Therefore, $\mathbb{R}_{nk}^{++}$ is the space of social state matrices and
we assume implicitly that all dimensions can be approximated quantitatively
(i.e. the model does not contemplate categorical data).

A vector $z \in \mathbb{R}_{k}^{+}$ describes the poverty thresholds for all dimensions, to
be interpreted as a parameter vector. We say that agent $i$ is poor with
respect to dimension $j$ if $y_{ij} \leq z_j$.

It will be assumed throughout that all individuals are homogeneous. That
is, agents have been normalized by some standard equivalence-scale proce-
dure. We allow, however, for the presence of dimensions of different relevance,
that will be scaled by a vector of weights $\beta \in \mathbb{R}_{k}^{++}$, with $\sum_{j \in K} \beta_{ij} = 1$. Those
weights may reflect the agents’ valuations or be an expression of the planner’s
priorities. Be as it may, we shall take that vector as a parameter externally
given.
Consider now the following:

**Definition 1:** A poverty evaluation problem, or simply a problem, is a point $\xi = (N, Y, z, \beta)$ in the space $N^n \times \mathbb{R}_{++}^n \times \mathbb{R}_{++}^k \times (0, 1)^k$, where $N^n$ is the set of all potential populations of size $n$.

A problem consists of a list of the agents, a social state matrix, a vector of poverty thresholds, and a vector of weights for the different poverty dimensions.

Let $D = \bigcup_{n,k \in \mathbb{N}} N^n \times \mathbb{R}_{++}^n \times \mathbb{R}_{++}^k \times (0, 1)^k$ denote the set of all possible problems. Then,

**Definition 2:** A poverty index is a function $P : D \to \mathbb{R}$.

A poverty index is a function that associates to each problem $\xi \in D$ a real number that summarizes the extent and intensity of poverty in society.

Let $Z = (z_1, z_2, \ldots, z_n)$ denote the social state matrix whose rows replicate the threshold vector $z$. Now define $\xi_0 := (N, Z, z, \beta)$, that is, $\xi_0$ is the problem in which all agents achievements are identical and correspond con the poverty threshold vector. The special case $\xi_0$ will play a relevant role in the ensuing discussion, as $P(\xi_0)$ will be a key reference value in the analysis of the incidence.

**Definition 3:** Given a problem $\xi$ and a poverty index, $P$, we define agent $i$’s individual poverty measure as follows: $P_i(\xi) = P ((\{i\}, y_i, z, \beta))$.

An agent’s poverty measure is simply the poverty index applied to a society consisting of a single agent. We shall say that $P_i(\xi_0)$ is agent $i$’s individual poverty threshold.

The set of the poor relative to a problem $\xi$ will be denoted by $N^Q(\xi) \subset N$, in the understanding that $y_i \leq z_i$, for all $i \in N$, implies $N^Q(\xi) = N$; and $y_i >> z_i$, for all $i \in N$, implies $N^Q(\xi) = \emptyset$.

The following three properties refer to the most basic features of a poverty index:

- **Monotonicity:** Let $\xi = (N, Y, z, \beta)$, $\xi' = (N, Y', z, \beta) \in D$, with $y_i' << y_i$, for all $i \in N^Q(\xi)$, $y_h' = y_h$, for all $h \notin N^Q(\xi)$. Then, $P(\xi') > P(\xi)$.

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1Given two vectors, $y, y' \in \mathbb{R}^n$, the expression $y >> y'$ means $y_i > y_i'$ for all $i$.

2Foster and Shorrocks (1991) consider five basic properties for a function to be considered a poverty index, the three listed below plus Symmetry and Replication Invariance. We do not require those properties for our discussion.
Monotonicity says that reducing the achievements of the poor, other things equal, will increase poverty. Note that we require $P$ to increase only if all the achievements of all the poor are reduced.

- **Focus:** Let $\xi = (N, Y, z, \beta), \tilde{\xi} = (\tilde{N}, \tilde{Y}, z, \beta)$ be two problems in $D$ such that $\tilde{y}_i = y_i$, for all $i \in N^Q(\xi)$, with $N^Q(\xi) = N^Q(\tilde{\xi})$. Then, $P(\xi) = P(\tilde{\xi})$.

The property of focus establishes that changes in the achievements of the non-poor, other things equal, should not affect the index.

The last property, restricted continuity, requires a piece of notation. For a given problem $\xi = (N, Y, z, \beta) \in D$ define $\phi_\xi(Y)$ as the set of problems $\tilde{\xi} \in D$ such that $\tilde{\xi} = (N, Y', z, \beta)$, with $N^Q(\tilde{\xi}) = N^Q(\xi)$. That is, all problems in $\phi_\xi(Y)$ are identical except for the value of the social state matrix. In particular all have the same number of agents, the same threshold vector, the same weighting system and the same set of the poor. Then,

- **Restricted continuity:** For all $\xi \in D$, $P$ is a continuous function of $Y$ on $\phi_\xi(Y)$.

### 2.2 Poverty indices and the incidence problem

Let us consider an example that shows that choosing the set of the poor independently on the poverty measure yields contradictory outcomes.

**Example**  Consider an evaluation problem with $n = k = 3$, $\beta_1 = \beta_2 = \beta_3$,

$$Y = \begin{pmatrix} 10 & 6 & 4 \\ 10 & 15 & 3.1 \\ 0.9 & 15 & 3.1 \end{pmatrix}$$

and $z = (1, 12, 3)$. According to the union approach, individuals 1 and 3 are poor while individual 2 is not. Let $P$ be a standard decomposable poverty index, given by:

$$P(\xi) = \frac{1}{nk} \sum_{i=1}^{q} \sum_{j=1}^{k} (1 - \frac{y_{ij}}{z_j})$$

(where $q$ stands for the number of poor). If we compute the poverty measure for this problem we get:

$$P(\xi) = \frac{1}{9} \left[ \sum_{k=1}^{3} (1 - \frac{y_{1k}}{z_k}) + \sum_{k=1}^{3} (1 - \frac{y_{3k}}{z_k}) \right] = -0.928$$
By monotonicity, which violates monotonicity.

Therefore, the interplay of an externally given set of poverty thresholds with an arbitrary way of defining the set of the poor may yield nonsensical outcomes. Indeed, it can be shown that assuming monotonicity, focus, and restricted continuity on a poverty index, implies determining endogenously the set of the poor. We shall show that next.

**Lemma 1** Let \( P \) be a poverty index that satisfies monotonicity, focus, and restricted continuity. Then:

(i) \( P(\xi) > P(\xi_0) \implies N^Q(\xi) \neq \emptyset \).

(ii) \( N^Q(\xi) \neq \emptyset \) implies that either \( P(\xi) > P(\xi_0) \) or \( P(\xi) = P(\xi_0) \) and \( \xi = \lim_{\nu \to \infty} \{\xi^\nu\} \), where \( \{\xi^\nu\} \) is a sequence of problems \( \xi^\nu = (N, Y^\nu, z, \beta) \), with \( P(\xi^\nu) > P(\xi_0) \) for all \( \nu \).

**Proof.** (i) Suppose, for the sake of contradiction, that \( P(\xi) > P(\xi_0) \) with \( N^Q(\xi) = \emptyset \). Focus implies that \( P(\xi) = a \) (constant) for all \( \xi \in D \) such that \( N^Q(\xi) = \emptyset \). Let now \( Y' \) be a social state matrix such that \( y'_i >> z \), for all \( i \in N \). Clearly, \( N^Q(N, Y', z, \beta) = \emptyset \). Moreover, by monotonicity, \( P(N, Y', z, \beta) \leq P(\xi_0) \). Therefore, we find that \( P(\xi) = a \leq P(\xi_0) \), for all \( \xi \in D \) such that \( N^Q(\xi) = \emptyset \), against the hypothesis.

(ii) Suppose, once more for the sake of contradiction, that \( N^Q(\xi) \neq \emptyset \) with \( P(\xi) < P(\xi_0) \), for some \( \xi \in D \). By restricted continuity and focus we can find \( Y' \) such that \( P(\xi) < P(\xi') < P(\xi_0) \), with \( y'_i << y_i \), for all \( i \in N^Q(\xi) \), \( y'_i = y_i \), for \( i \in N \setminus N^Q(\xi) \). Let now \( Y'' \) be such that \( y''_i = z \) for all \( i \in N^Q(\xi) \), and \( y''_i = y_i \), otherwise. Focus requires that \( P(\xi'') = P(\xi_0) \), which violates monotonicity.

(iib) Let \( N^Q(\xi) \neq \emptyset \) with \( P(\xi) = P(\xi_0) \). For all \( i \in N^Q(\xi) \) substitute \( y_i \) by \( y'_i \). Call \( Y' \) to the new social matrix and \( \xi' \) to the resulting problem. By monotonicity, \( P(\xi') > P(\xi_0) \). Let now \( Y(\lambda) = \lambda Y + (1 - \lambda) Y' \). As \( \lambda \) varies from 0 to 1 it generates a sequence of problems that converge to \( \xi \) with the required properties.

Lemma 1 says, approximately, that the set of the poor is nonempty if and only if the poverty index exceeds the value of the threshold problem \( \xi_0 = (N, Z, z, \beta) \). It is not an exact "if and only if" statement as we should admit limit cases in which \( P(N, Y, z, \beta) = P(\xi_0) \), where \( Y = \lim_{\nu \to \infty} Y^\nu \), with \( P(N, Y^\nu, z, \beta) > P(\xi_0) \), for all \( \nu \), given the way of defining the poor
Lemma 2 Let $P$ be a poverty index that satisfies monotonicity, focus and restricted continuity. Then:
(i) $P_i(\xi) > P_i(\xi_0) \implies i \in N^Q(\xi)$. 
(ii) $i \in N^Q(\xi)$ implies $P_i(\xi) > P_i(\xi_0)$ or $P_i(\xi) = P_i(\xi_0)$ with $y_i = \lim_{\nu \to \infty} \{y_i^\nu\}$, and $P(\{i\}, y_i^\nu, z, \beta) > P_i(\xi_0)$ for all $\nu$.

Proof. (i) If the result were not true, we could find $h \notin N^Q(\xi)$ with $P_i(\xi) > P_i(\xi_0)$. But this contradicts (i) in Lemma 1 for problem $\xi_i = [\{i\}, y_i, z, \beta]$, as $N^Q(\xi_i) = \emptyset$.

(ii) If the result were not true, we could find some $i \in N^Q(\xi)$ such that $P_i(\xi) < P_i(\xi_0)$. But this contradicts (iia) in Lemma 1 for problem $\xi_i = [\{i\}, y_i, z, \beta]$, as $N^Q(\xi_i) \neq \emptyset$.

(iib) If the result were not true, we could find some $i \in N^Q(\xi)$ such that $P_i(\xi) = P_i(\xi_0)$ but there is no sequence $\{y_i^\nu\}$ converging to $y_i$, with $P(\{i\}, y_i^\nu, z, \beta) > P_i(\xi_0)$ for all $\nu$. But this contradicts (iib) in Lemma 1 for problem $\xi_i = [\{i\}, y_i, z, \beta]$, as $N^Q(\xi_i) \neq \emptyset$.

Those results show that the most elementary axioms on poverty measures have definite implications on the poverty incidence (which should not be that surprising). The set of the poor consists, precisely, of all those agents whose individual poverty measures exceed their corresponding individual poverty thresholds, or match that value as limit points of a sequence of points that exceed that value. The properties of monotonicity, focus, and restricted continuity, therefore, determine completely the set of the poor.

We can summarize the discussion in terms of the following result:

Theorem: Let $P$ be a poverty measure that satisfies monotonicity, focus and restricted continuity. Then, for each problem $\xi \in D$, the set of the poor, $N^Q(\xi)$, consists exactly of those agents $i \in N$ whose vector of achievements $y_i$ is such that $P_i(\xi) \geq P_i(\xi_0)$, and $y_i = \lim_{\nu \to \infty} \{y_i^\nu\}$ with $P(\{i\}, y_i^\nu, z, \beta) > P_i(\xi_0)$.

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3 As we established that agent $i$ is poor with respect to dimension $j$ when $y_{ij} \leq z_j$, it follows that all agents are poor when $Y = Z$. Therefore, the value $P(\xi_0)$ does not exclude the presence of poverty.
3 A final comment

We have shown that in a multidimensional context standard poverty indices fully determine who are the poor. That is, when $n \geq 2$ deciding separately about the incidence and the intensity of poverty is not appropriate, contrary to the standard usage. This does not invalidate the design of those indices but rather adds more relevance to their characterizations, as they also imply the definition of who is poor in a single shot.

The case $n = 1$ trivially satisfies our theorem. It is a special case in the sense that setting the threshold determines directly the set of the poor. Then, when we introduce a poverty measure that satisfies monotonicity, focus and restricted continuity, we fix the limit value of the index that separates the poor from the rest. For $n \geq 2$ we can consider different sets of the poor if no restriction is imposed on the poverty index. Yet when we assume those three basic properties, the poverty measure determines the incidence as well as the intensity of the poverty and there is no degree of freedom left.

References


